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THE METHOD OF GENERALIZED FUNCTIONS IN BOUNDARY-VALUE PROBLEMS OF COUPLED THERMOELASTODYNAMICS†

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Using the theory of generalized functions, the method of boundary integral equations is developed to solve four non-stationary boundary-value problems of coupled thermoelastodynamics for media with anisotropy of the elastic properties and thermal isotropy. Regular integral representations of solutions in Laplace transform space with respect to time and singular boundary integral equations are constructed. The uniqueness of the solution of the boundary-value problems, including those in the shock-wave class, is proved. © 2001 Elsevier Science Ltd. All rights reserved.

Solutions of boundary-value problems of thermoelasticity in regions of classical forms, obtained by the method of complete and incomplete separation of variables, have been investigated most thoroughly, whereas the dynamics of thermoelastic bodies and media with a complex geometry of boundaries have been much less investigated. To solve problems concerning the stress concentration in a homogeneous and piecewise-homogeneous linear medium with stress concentrators of different forms, the boundary integral equation (BIE) method is effective. It was developed by a number of researchers to solve static and quasi-static problems of thermoelasticity in regions with complex geometry, and has also been used to solve boundary-value problems of uncoupled thermoelastodynamics with specified unsteady heat fluxes on the boundary [1, 2]. Unlike these studies, in the present paper a BIE method is developed for solving non-stationary boundary-value problems for a model of a coupled thermoelastic medium. The use of such a model, in which the temperature gradient influences the deformation of the medium while the rate of volume deformation affects the change in temperature, is necessary to calculate the influence of dynamic loads on the thermal stressed state.

A BIE method for solving boundary-value problems of coupled thermoelastodynamics was considered earlier in [3–5]. In this case, the traditional approach was used to construct the governing relations for the displacements and temperature and the BIEs, which is based on the identities of Betti reciprocity and their analogues for a thermoelastic medium. In the present paper, a new approach is proposed in developing the BIE method, based on the use of the theory of generalized functions, following a procedure proposed earlier [6]. It enables one, comparatively simply, to introduce into consideration classes of derivative-discontinuous solutions (shock waves), to derive the conditions at the wave fronts and to construct dynamic analogues of the Somigliana and Gauss formulae for a thermoelastic medium in generalized function space.

1. THE GOVERNING RELATIONS SHOCK WAVES

A linear isotropic thermoelastic medium is characterized by a finite number of thermodynamic parameters: the mass density ρ, the Lamé constants of elasticity, λ and μ and the thermoelastic constants γ, η and $\varkappa$. In a Cartesian system of coordinates, such a medium is described by the following system of equations [7]

$$\sigma_{ij,j} - \rho_{iii} + F_i = 0$$

$$\theta_{,jj} - \varkappa^{-1}\theta - \eta_{iij,j} + F_{N+1} = 0, \quad j = 1, \ \dots, N \tag{1.1}$$

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where u_1 are the components of the displacement vector $u(x, t)$, $u_{i,j} = \partial u_i/\partial x_j = \partial_j u_i$, $\theta \equiv u_{N+1}$ is the relative temperature ($\theta = T(x, t) - T(x, 0)$), T is the absolute temperature, F_i are the components of the mass force, $F_{N+1} = (\lambda_0\kappa)^{-1}W$, W is the amount of heat released per unit volume in unit time, λ_0 is the thermal conductivity and σ_{ij} are the components of the stress tensor related to the displacements by the Duhamel–Neyman laws

$$\sigma_{ij} = \lambda\delta_{ij}\delta_{kl}u_{k,l} + \mu(u_{i,j} + u_{j,i}) - \gamma\delta_{ij}\theta \tag{1.2}$$

When $N = 2$ the case of plane deformation is considered, and when $N = 3$ the case of spatial deformation. Everywhere, summation is carried out over repeated subscripts within their given range of variation.

Substituting expression (1.2) into system (1.1), we obtain a complete system of equations in u

$$\begin{aligned}
&L_{ij}(\partial_x, \partial_t)u_j + F_i = 0\\
&L_{ij} = (\lambda+\mu)\partial_i\partial_j + (\mu\Delta - \rho\partial_t\partial_t)\delta_{ij} - \gamma\delta_{j(N+1)}\partial_i, \quad i = 1, 2, \dots, N\\
&L_{(N+1)j} = (\Delta - \varkappa^{-1}\partial_t)\delta_{j(N+1)} - \eta(1 - \delta_{j(N+1)})\partial_t\partial_j, \quad j = 1, 2, \dots, N+1
\end{aligned} \tag{1.3}$$

This is a system of the mixed hyperbolic–parabolic type. Waves propagating in the thermoelastic medium may be shock waves. The equation of the wave front F has the form

$$\det\{L_{ij}^2(\nu, \nu_t)\} = \det\{L_{ij}^e(\nu, \nu_t)\}\,\|\nu\|^2, \quad \|\nu\|^2 = \sum_{i=1}^{N}\nu_i^2 \tag{1.4}$$

where L_{ij}^2 is the principal part of the operator $L_{ij}(\partial_x, \partial_t)$, which contains only the higher second-order derivatives, and L_{ij}^e is the differential operator of the equations of motion of the elastic solid with the parameters λ, μ and ρ; (ν, ν_t) and $(\nu_1, \dots, \nu_N, \nu_t)$ is the vector of the normal to F in R^{N+1}.

From (1.4) it follows that

$$\text{either } \|\nu\| = 0, \quad \text{or} \quad \det\{L_{ij}^e(\nu, \nu_t)\} = 0$$

The first equation describes the characteristic surface of the classic parabolic equation, which has the form t = const and does not determine the wave front in R^N. The second equation describes the wave fronts F_t moving in R^N with velocity

$$c = |\nu_t|/\|\nu\|, \quad c = c_j, \quad j = 1, 2 \tag{1.5}$$

where $c_1 = \sqrt{(\lambda + 2\mu)/\rho}$ is the velocity of the elastic body waves and $c_2 = \sqrt{\mu/\rho}$ is the velocity of the shear waves. Thus, wave fronts (shock waves) in the thermoelastic medium travel at the velocity of elastic waves. To derive the conditions at the wave fronts, it is convenient to use the theory of generalized functions.

We introduce a space of finite infinitely differentiable vector functions

$$\varphi(x, t) = \{\varphi_1(x, t), \dots, \varphi_{N+1}(x, t)\} \in D_{N+1}(R^{N+1})$$

The conjugate space $D'_{N+1}(R^{N+1})$ – the space of generalized vector functions $\hat{f}(x, t) = \{\hat{f}_i(x, t),$ $i = 1, 2, \dots, N + 1\}$ – of linear functionals on $D_{N+1}(R^{N+1})$ corresponds to this. Taking the rules for the differentiation of generalized functions into account, we obtain the equations of motion of $D'_{N+1}(R^{N+1})$

$$\begin{aligned}
&\hat{\sigma}_{ij,j} - \rho\hat{u}_{i,tt} + \rho\hat{G}_i = \lfloor \sigma_{ij}\nu_j - \rho u_{i,t}\nu_t \rfloor_F \delta_F(x, t) +\\
&+ \partial_j\Big(\lfloor (\lambda u_k\nu_k - \gamma\theta)\delta_{ij} + \mu(u_i\nu_j + u_j\nu_i) \rfloor_F \delta_F(x, t)\Big) - \rho\partial_t([u_i]_F\nu_t\delta_F(x, t))\\
&\hat{\theta}_{,jj} - \varkappa^{-1}\hat{\theta}_{,t} - \eta\hat{u}_{j,jt} + \varkappa^{-1}\hat{\theta} = \lfloor (\theta_{,j} - \eta\dot{u}_j)\nu_j - \varkappa^{-1}\theta\nu_t \rfloor_F \delta_F(x, t) +\\
&+ \partial_j\Big(\lfloor \theta\nu_j - \eta u_j\nu_t \rfloor_F \delta_F(x, t)\Big)
\end{aligned}$$

The square brackets denote a sudden change in the function on the characteristic surface F in R^{N+1} corresponding to the wave front F_t in R^N. For the conditions of continuity of the medium to be conserved, and for the vector function u to be the solution of system (1.1) in $D'_{N+1}(R^{N+1})$, the sudden changes must satisfy the following conditions

$$\lfloor u_j \rfloor_F = 0, \quad [\theta]_F = 0$$

$$\lfloor \sigma_{ij}\nu_j - \rho\nu_t\dot{u}_i \rfloor_F = 0, \quad \lfloor (\theta_{,j} - \eta\dot{u}_j)\nu_j - \varkappa^{-1}\theta\nu_t \rfloor_F = 0; \quad i,\ j = 1,\ 2,\ \ldots,\ N \tag{1.6}$$

Taking Eqs (1.3) and the equality $n_j = \nu_j/\|\nu\|$ into account, we derive from them the laws of conservation on the moving wave fronts F_t in R^N

$$[u]_{F_t} = 0, \quad [\theta]_{F_t} = 0 \tag{1.7}$$

$$\lfloor \sigma_{ij}n_j + \rho c\dot{u}_i \rfloor_{F_t} = 0 \tag{1.8}$$

$$\lfloor \theta_{,j}n_{,j} \rfloor_{F_t} = \eta\lfloor \dot{u}_j n_j \rfloor_{F_t} \tag{1.9}$$

where n_j are the components of the wave vector, i.e. the unit vector perpendicular to F_t and in the direction of propagation of the wave. The first equality of (1.7) is the condition of conservation of continuity of the medium, and (1.8) is identical with the well-known law of conservation of momentum on the wave fronts of shock waves in elastic media. It follows from the second relation of (1.7) and (1.9) that, on the wave fronts, the temperature is continuous, but its gradient undergoes a sudden change proportional to the sudden change in the normal component to the displacement velocity front of the medium.

By virtue of the continuity of u, the first equality of (1.7) implies the condition that the tangential derivatives to the front are equal, which has the form

$$\lfloor \nu_j\dot{u}_i - \nu_t u_{i,j} \rfloor = 0, \quad i = 1,\ 2,\ \ldots,\ N+1, \quad j = 1,\ 2,\ \ldots,\ N \tag{1.10}$$

since the vectors $\tau^i = (\delta_1^i\nu_t, \ldots, \delta_N^i\nu_t, -\nu_i)$ lie in the tangential plane to F:

$$\sum_{j=1}^{N+1} \tau_j^i\nu_j = \sum_{j=1}^{N} \delta_j^i\nu_t\nu_j - \nu_i\nu_t = \nu_i\nu_t - \nu_i\nu_t = 0$$

We will call the solution of Eqs (1.1) that satisfies conditions (1.7)–(1.9) the classical solution.

2. GREEN'S TENSOR AND ITS PROPERTIES

We will consider the fundamental solutions of Eqs (1.1). Among them, Green's tensor, corresponding to $F_i = \delta(x,t)\delta_{i[j]}$ ($\delta(x,t) \equiv \delta(x)\delta(t)$, δ_{ij} is the Kronecker delta) occupies a special place and satisfies the initial conditions

$$U_i^j(x,\ 0) = 0, \quad i,\ j = 1,\ 2,\ \ldots,\ N+1; \quad \dot{U}_i^j(x,\ 0) = 0, \quad j = 1,\ 2,\ \ldots,\ N, \quad x \neq 0 \tag{2.1}$$

If the mass forces and heat sources are known, then, for an unbounded medium, the solution has the form of the convolution

$$u_i(x,\ t) = U_i^j(x,\ t) * F_j(x,\ t) = \int_0^\infty d\tau \int_{R_N} U_i^j(x-y,\ t-\tau)F_j(y,\ \tau)dV(y) \tag{2.2}$$

$$i, j = 1, 2, \ldots, N+1$$

The Laplace transform with respect to time was constructed [8] for this tensor $\overline{U}_i^j(x,p)$ (p is the Laplace transformation parameter) for $N = 2, 3$. Construction of the analytical formula for the original is impossible, and therefore the BIEs for solving the boundary-value problems are constructed for the displacement transform.

Consider the properties of $\bar{U}_i^j(x, p)$.

Properties of symmetry

$$\bar{U}_i^j(x,\ p) = \bar{U}_j^i(x,\ p),\quad \bar{U}_i^j(-x,\ p) = \bar{U}_i^j(x,\ p)$$

$$\bar{U}_i^3(x,\ p) = \frac{m(\lambda+2\mu)}{\eta p\varkappa}\bar{U}_3^i(x,\ p),\quad \bar{U}_i^{N+1}(-x,\ p) = -\bar{U}_i^{N+1}(x,\ p) \tag{2.3}$$

$$\bar{U}_{N+1}^i(-x,\ p) = -\bar{U}_{N+1}^i(x,\ p),\quad \bar{U}_{N+1}^{N+1}(-x,\ p) = \bar{U}_{N+1}^{N+1}(x,\ p);\quad i,\ j = 1,\ 2,\ \ldots,\ N$$

Unlike an elastic medium, here Green's tensor, generally speaking, is asymmetrical both with respect to the indices and the argument.

The asymptotic form at infinity. For $N = 2$ [8]

$$\lim_{\|x\|\to\infty} \bar{U}_i^k(r,\ p) = 0 \tag{2.4}$$

since $\bar{U}_i^k(r,\ p)$ is expressed in terms of the MacDonald function $K_n(z)$, while $K_n(z) \to 0$ when $z \to \infty$, $\mathrm{Re}\, z > 0$.

For $N = 3$, the damping is exponential since $\exp(-\zeta_j r)$, $\mathrm{Re}\zeta_j > 0$ occurs in the dynamic functions.

To construct the asymptotic form of $\bar{U}_i^k(r, p)$ for large t, since

$$\lim_{t\to\infty} U_i^j(x,\ t) = \lim_{p\to+0} p\bar{U}_i^j(x,\ p)$$

we consider the properties of the quantity $p\bar{U}_i^j(x, p)$. When $p \to +0$ ($\mathrm{Im}\, p = 0$), we have $\zeta_2^2 \to 0$, $p \ln p \to 0$.

Then, for $N = 2$

$$p\bar{U}_i^k(x,\ p) \sim -\frac{p\ln p}{2\pi\mu}\left[\delta_i^k + \frac{1+\varepsilon}{2}\ \frac{c_2^2(2+\varepsilon) - 2c_1^2(1+\varepsilon)}{c_1^2(1+\varepsilon)^2 - pk} r_{,i}r_{,k}\right] \to 0$$

$$p\bar{U}_3^k(x,\ p) \sim -\frac{\eta p^2 r}{8\pi(\lambda+2\mu)}P(p)r_{,k} \to 0,\quad p\bar{U}_i^3(x,\ p) \sim -\frac{mpr}{8\pi\varkappa}P(p)r_{,i} \to 0$$

$$p\bar{U}_3^3(x,\ p) \sim -\frac{p}{4\pi\varkappa}P(p) \to 0,\quad P(p) = \frac{c_1^2(1+\varepsilon)^2 \ln p}{c_1^2(1+\varepsilon)^2 - p\varkappa}$$

i.e. a disturbance at any point of the medium attenuates with time.

Asymptotic representations of $U_i^k(r,\ p)$ when $x \to 0$. Using asymptotic representations for special functions, the asymptotic forms of $\bar{U}_i^k$, presented in Table 1, were plotted. From formula (2.2), the thermal stressed state of the medium when acted on by a pulsed concentrated and time-distributed force and heat sources for rock was calculated.†

Table 1

	$N = 2\ (i, k = 1, 2)$	$N = 3\ (i, k = 1, 2, 3)$
$\bar{U}_i^k$	$(4\pi\mu)^{-1}\{(1-c^2)r_{,i}r_{,k} - \ln r\delta_i^k(1+c^2)\}$	$(8\pi\mu r)^{-1}\{(1-c^2)\ r_{,i}r_{,k} + \delta_i^k(1+c^2)\}$
$\bar{U}_i^{N+1}$	$-(4\pi\varkappa)^{-1}mr_{,i}r\ln r$	$(8\pi\varkappa)^{-1}mr_{,i}$
$\bar{U}_{N+1}^k$	$-(4\pi(\lambda+2\mu))^{-1}\eta pr_{,k}r\ln r$	$(8\pi(\lambda+2\mu))^{-1}\eta pr_{,k}$
$\bar{U}_{N+1}^{N+1}$	$-(2\pi\varkappa)^{-1}\ln r$	$(4\pi\varkappa r)^{-1}$

†KUPESIVA, B.N., Fundamental solutions and boundary integral equations of problems of coupled thermoelastodynamics. Candidate dissertation 01.02.04, almaty, 1998.

3. FORMULATION OF THE BOUNDARY-VALUE PROBLEMS AND THE UNIQUENESS OF THE SOLUTIONS

Suppose the medium occupies a region S^- bounded by a closed Lyapunov surface S with an outward normal n. The initial and boundary conditions are known, namely,

$$u_i(x,\ 0)=u_i^0(x),\ \ \theta(x,\ 0)=\theta^0(x),\ \ x\in(S^-+S);\ \ \dot{u}_i(x,\ 0)=\dot{u}_i^0(x),\ \ x\in S^-$$
$$u^0\in C(S+S^-),\ \ \dot{u}^0\in C(S+S^-) \tag{3.1}$$

Problem 1. On the boundary ($x\in S$), the load and heat flux are specified:

$$\sigma_{ij}(x,\ t)n_j(x)=q_i^S(x,\ t),\quad q_i^S\in C'(S\times[0,\ \infty)) \tag{3.2}$$

$$\partial\theta(x,\ t)/\partial n=q^S(x,\ t),\quad q^S\in C(S\times[0,\ \infty)) \tag{3.3}$$

Problem 2. On the boundary ($x\in S$), the displacements and temperature are specified:

$$u_i(x,\ t)=u_i^S(x,\ t),\ \ u_i^S(x,\ 0)=u_i^0(x);\ \ u_i^S\in C(S\times[0,\ \infty)) \tag{3.4}$$

$$\theta(x,\ t)=\theta^S(x,\ t),\ \ \theta^S(x,\ 0)=\theta^0(x):\ \theta^S\in C(S\times[0,\ \infty)) \tag{3.5}$$

Problem 3. On the boundary ($x\in S$), the displacements (conditions (3.4)) and heat flux (conditions (3.3)) are specified.

Problem 4. On the boundary ($x\in S$), the load (conditions (3.2)) and temperature (conditions (3.5)) are specified.

Here, $C(\ldots)$ is a class of continuous functions on the given set, and $C'(\ldots)$ is a class of piecewise-continuous bounded functions. On the fronts of the solutions, the conditions of continuity (1.7)–(1.9) are satisfied.

It is required to construct the governing relations and BIEs for these problems.

Further, to simplify the calculations, it is convenient to represent σ_{ij} in the form

$$\sigma_{ij}=C_{ij}^{kl}u_{k,l}-\gamma\delta_{ij}\theta \tag{3.6}$$

where C_{ij}^{kl} are the tensor components of the constants of elasticity, generally speaking, of an anisotropic thermoelastic medium, satisfying the symmetry conditions

$$C_{ij}^{kl}=C_{ji}^{kl}=C_{ij}^{lk}=C_{kl}^{ij} \tag{3.7}$$

In the isotropic case it has the form

$$C_{ij}^{kl}=\lambda\delta_{ij}\delta_{kl}+\mu(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})$$

Theorem 3.1. If a solution of the boundary-value problem exists, it is unique.

Proof. We put

$$2W(u,\ \theta,\ t)=\sigma_{ij}u_{i,j}+\rho\|\dot{u}\|^2+\gamma\theta u_{j,j}+\gamma(\eta\varkappa)^{-1}\theta^2=$$
$$=C_{ij}^{kl}u_{i,j}u_{k,l}+\rho\|\dot{u}\|^2+\gamma(\eta\varkappa)^{-1}\theta^2,\quad i,j=1,\ 2,\ \ldots,\ N$$

We multiply the first equation of system (1.3) scalarly by $\dot{u}_i$, and the second by $\gamma\eta^{-1}\theta$, and add. Grouping terms and using the equality

$$\dot{u}_{i,j}\sigma_{ij}=\frac{1}{2}(C_{ij}^{kl}u_{i,j}u_{k,l})_{,t}-\gamma\theta\dot{u}_{i,j}$$

we obtain

$$(\dot{u}_i\sigma_{ij}+\gamma\eta^{-1}\theta\theta_{,j})_{,j}-W_{,t}-\gamma\eta^{-1}\|\mathrm{grad}\,\theta\|^2+\dot{u}_jF_j+\gamma\varkappa\eta^{-1}\theta F_{N+1}=0$$

We integrate with respect to $S^- \times (0, t)$ using Gauss' theorem. We have

$$\int_0^t dt \int_S (\dot{u}_{i,j}\sigma_{ij} + \gamma\eta^{-1}\theta\theta_{,j})n_j dS(y) - \int_{S^-} (W(u, \theta, t) - W(u, \theta\theta))dV(x) -$$

$$-\gamma\eta^{-1}\int_0^t dt \int_{S^-} \|\mathrm{grad}\,\theta\|^2 dV(x) + \sum_F \int_F \lfloor(\dot{u}_i\sigma_{ij} + \gamma\eta^{-1}\theta\theta_{,j})\nu_j - W(u, \theta, t)\nu_t\rfloor_F dF(x) +$$

$$+\int_0^t dt \int_{S^-} (\dot{u}_j F_j + \gamma\varkappa\eta^{-1}\theta F_{N+1})dV(x) = 0$$

where dF is the differential of the area of the characteristic surface F in $R^N \times t$, which corresponds to the wave front F_t in R^N. We integrate over the surfaces of all the wave fronts entering the region of integration. We evaluate the integral obtained. Using the conditions on the fronts (1.7)–(1.10), we transform

$$(\dot{u}_i\sigma_{ij} + \gamma\eta^{-1}\theta\theta_{,j})\nu_j - W(u,\theta,t)\gamma_t = \frac{1}{2}[\dot{u}_i(\sigma_{ij}\nu_j - \rho\dot{u}_i\nu_t)] - \frac{1}{2}[\sigma_{ij}(\nu_t u_{i,j} - \dot{u}_i\nu_j)] +$$

$$+\gamma\eta^{-1}\left[\theta\theta_{,j}\nu_j - \frac{1}{2}\nu_t(\eta\theta u_{j,j} + \varkappa^{-1}\theta^2)\right] = \frac{1}{2}\dot{u}_i^-[\sigma_{ij}\nu_j - \rho\dot{u}_i\nu_t]\dot{u}_i^- + \frac{1}{2}\sigma_{ij}^-\nu_j[\dot{u}_i] -$$

$$-\frac{1}{2}\nu_t u_{i,j}^- C_{ij}^{kl}[u_{k,l}] + \frac{1}{2}\nu_t u_{i,j}^-\gamma[\theta]\delta_{ij} + \gamma\eta^{-1}\theta[\theta_{,j}\nu_j] - \frac{1}{2}\gamma\theta\nu_t[u_{j,j}] =$$

$$= \gamma\eta^{-1}\theta[(\theta_{,j} - \eta\dot{u}_j)\nu_j + \eta(\dot{u}_j\nu_j - \nu_t u_{j,j})] = 0$$

Finally, we obtain

$$\int_0^t dt \int_S (\dot{u}_{i,j}\sigma_{ij} + \gamma\eta^{-1}\theta\theta_{,j})n_j dS(y) + \int_0^t dt \int_{S^-} (F_j\dot{u}_j + \gamma\eta^{-1}\varkappa\theta F_{N+1})dV(x) =$$

$$= \int_{S^-} (W(u,\theta,t) - W(u,\theta,0))dV(x) + \int_0^t dt \int_{S^-} \|\,\mathrm{grad}\,\theta\,\|^2\, dV(x)$$

This equality expresses the law of conservation of energy. The uniqueness of the solution of the boundary-value problem follows from this.

In fact, we will assume that two solutions $u^k(x, t)$ $(k = 1, 2)$ exist. Then the difference $u = u^1 - u^2$ satisfies the zero boundary and initial conditions and the corresponding functions $F_i = 0$ and $Q = 0$. For u we have $\int_{S^-} W(u, \theta, t)dV(x) + \int_0^t dt \int_{S^-} \|\mathrm{grad}\theta\|^2\, dV(x) = 0$. Since the integrand is positive, it follows that $u \equiv 0$, i.e. $u^1 = u^2$.

We will now formulate the problems in Laplace transform space with respect to time, since the kernels of the BIEs can be constructed only in this space. The equation of motion (1.1) in this space, taking into account the conditions on the fronts, take the form

$$\begin{aligned}
&\overline{\sigma}_{ij,j} - \rho p^2\overline{u}_i + \rho\overline{F}_i + \overline{G}_i = 0\\
&\overline{\theta}_{,jj} - p\varkappa^{-1}\overline{\theta} - \eta p\overline{u}_{j,j} + \overline{F}_{N+1} + \overline{G}_{N+1} = 0\\
&\overline{G}_i = (\rho\dot{u}_i^0(x) + \rho p u_i^0(x))H_S^-(x)\\
&\overline{G}_{N+1} = (\varkappa^{-1}\theta^0(x) - \eta u_{j,j}^0)H_S^-(x) + \eta u_j^0 n_j \delta_S(x)
\end{aligned} \tag{3.8}$$

In Laplace transform space the initial conditions are transformed into asymptotic conditions

$$\lim_{p\to+\infty} p\overline{u}(x,p) = u^0(x), \quad \lim_{p\to+\infty} p\overline{\theta}(x,p) \to \theta^0(x), \quad x \in (S^- + S)$$

$$\lim_{p\to+\infty} p^2\overline{u}(x,p) = \dot{u}^0(x), \quad x \in S^- \tag{3.9}$$

The boundary conditions have a form similar to (3.2)–(3.5), but only for the Laplace tranform.
We then construct the governing relations and boundary integral equations using the theory of generalized functions, following the procedure described in [6].

4. THE ANALOGUE OF THE SOMIGLIANA FORMULA IN GENERALIZED FUNCTION SPACE

We will use $H_S^{-1}(x)$ to denote the characteristic function of the set S^-, which for a smooth surface S has the form

$$H_S^-(x)=\begin{cases}1, & x\in S^-\\ \tfrac{1}{2}, & x\in S\\ 0, & x\notin S^-+S\end{cases} \tag{4.1}$$

We will extend the definition of the functions specified on S^- to R^N by introducing the functions

$$\hat{u}(x,p)=\bar{u}(x,p)H_S^-(x),\quad \hat{F}(x,p)=\bar{F}H_S^-(x)$$

which will be regarded as generalized functions from $D'_{N+1}(R^N)$. Note that

$$\partial_j\hat{f}=H_S^-(x)\partial_j f-n_j f\delta_S(x) \tag{4.2}$$

where $n_jf\delta_s(x)$ – a singular generalized function – is a simple layer on S. Without loss of generality, we will assume that the point $x = 0$ belongs to S^-.

Theorem 4.1. If $u(x, p)$ is the classical solution of the boundary-value problem, we have

$$\hat{u}_m(x,p)=\bar{U}_m^i*(\rho\hat{F}_i+\hat{G}_i)+\bar{U}_m^i*\bar{q}_i^S\delta_S(x)+C_{ij}^{kl}\bar{U}_{m,j}^i*(\bar{u}_k^S n_l\delta_S(x))+$$
$$+\bar{U}_m^{N+1}*(\varkappa^{-1}\hat{F}_{N+1}+\hat{G}_{N+1})+\bar{U}_m^{N+1}*(\bar{q}^S-\eta p\bar{u}_j^S n_j)\delta_S(x)+\bar{U}_{m,j}^{N+1}*(\bar{\theta}^S n_j\delta_S)$$
$$i,j,k,l=1,2,\ldots,N;\ m=1,2,\ldots,N+1$$

Proof. Using the rules for the differentiation of generalized functions and relation (4.2), we obtain equations for $\hat{u}(x,p)$ in $D'_N(R^N)$

$$L_{ij}(\partial_x,p)\hat{u}_j+\Psi_j=0,\quad i,j=1,2,\ldots,N+1 \tag{4.3}$$

$$\bar{\Psi}_i=\hat{F}_i+\hat{G}_i+n_j\bar{\sigma}_{ij}\delta_S(x)+C_{ij}^{kl}(\bar{u}_k n_l\delta_S(x))_{,j},\quad i=1,2,\ldots,N \tag{4.4}$$
$$\bar{\Psi}_{N+1}=\hat{F}_{N+1}+\hat{G}_{N+1}+(\bar{\theta}_{,j}-\eta p\bar{u}_j)n_j\delta_S(x)+(\bar{\theta}n_j\delta_S)_{,j}$$

Using the property of Green's tensor U_i^j, we consider the generalized solution of Eqs (4.3), which is represented in the form of the convolution

$$\hat{w}_i=\bar{U}_i^j*\hat{\Psi}_j,\quad i,j=1,2,\ldots,N+1 \tag{4.5}$$

Suppose supp $\varphi\notin S+S^-$, then

$$(U_i^j*\hat{\Psi}_j,\varphi_i)=(U_i^j*L_{jk}\hat{u}_k,\varphi_i)=(L_{jk}U_i^j*\hat{u}_k,\varphi_i)=(\delta_{ik}\delta(x)*\hat{u}_k,\varphi_i)=(\hat{u}_i,\varphi_i)=0 \tag{4.6}$$

i.e. $\hat{w} = 0$ for $x\notin S+S^-$. Consequently, $(\hat{w}-\hat{u})$, the solution of the homogeneous boundary-value problem (with zero boundary conditions and right-hand side of the equations), is also equal to zero for $x\notin S^-$. Hence, by virtue of the regularity of $(\hat{w}-\hat{u})$ and the uniqueness of the solution of the boundary-value problem, it follows that $\hat{w}=\hat{u}$.

Substituting expression (4.4) into (4.6) and using the rules for the differentiation of the convolution, taking into account the notation introduced, we obtain the formula indicated in the theorem. All the convolutions exist by virtue of the boundedness of S. The theorem is proved.

The formula of the theorem expresses the displacements and the temperature within the region in terms of their boundary values, and also the values of the load and heat flux on the boundary. It is an analogue of Somigliana's formula of the static theory of elasticity, which expresses the displacements of the elastic medium in terms of the boundary values of the load and displacements.

Since only some of the boundary functions are known, the formula does not give solutions of the boundary-value problem. One day we shall show that it also holds for $x \in S$ in the sense of determining $H_S^-(x)$ and gives boundary integral equations for solving the boundary-value problems in question.

5. TENSORS WITH COMPONENTS T_i^j AND G_i^j AND THEIR PROPERTIES

To write formula (4.5) in a convenient integral form, we will introduce a tensor with components $T_i^j(x, n, t)(i, j, k, l = 1, 2, \ldots, N, m = 1, 2, \ldots, N + 1)$

$$\overline{T}_m^k(x,n,p) = C_{ij}^{kl}\overline{U}_{m,j}^i n_l - \eta p \overline{U}_m^{N+1} n_k, \quad \overline{T}_m^{N+1} = \overline{U}_{m,j}^{N+1} n_j \tag{5.1}$$

In the isotropic case, a tensor with components $\overline{T}_i^k(x, p)$ was obtained in the dissertation mentioned in the earlier footnote.

Theorem 5.1. The tensor with components $\overline{T}_m^k$ with fixed k is the fundamental solution of Eqs (3.6) and (1.1), corresponding to a concentrated mass force and a heat source of the multipole type

$$\overline{F}_S = (C_{sj}^{[k]l} n_l \partial_j - \eta p n_k \delta_S^{N+1})\delta(x), \quad k = 1,2,\ldots,N$$

$$\overline{F}_S = \delta_S^{[N+1]} n_j \delta_{,j}, \quad k = N+1$$

Proof. We fix k. We act on $\overline{T}_m^k$ $(k, m = 1, 2, \ldots, N + 1)$ with the operator $L_{ij}(\partial_x, p)$ and use the equation for $\overline{U}_m^k$. We obtain

for $k = 1, 2, \ldots, N$

$$L_{sm}(\partial_x, p)T_m^k = \sum_{m=1}^{N+1} L_{sm}C_{ij}^{kl}U_{m,j}^i n_l - L_{sm}\eta p n_k U_m^{N+1} = \sum_{m=1}^{N+1} n_l C_{ij}^{kl} L_{sm} U_{m,j}^i - \eta p n_k L_{sm} U_m^{N+1} =$$

$$= -\sum_{m=1}^{N+1} n_l C_{ij}^{kl}\delta_S^i \delta_{,j}(x) + \eta p n_k \delta_S^{N+1}\delta(x)$$

for $k = N + 1$

$$L_{sm}(\partial_x, p)T_m^{N+1} = \sum_{m=1}^{N+1} L_{sm} n_j \partial_j U_m^{N+1} = -\delta_S^{N+1} n_j \delta_{,j}$$

$$s = 1, 2, \ldots, N+1, \quad j, l = 1, 2, \ldots, N$$

The theorem is proved.

We introduce tensors with components $\overline{\Sigma}_{ij}^m$ and $\overline{G}_i^m$ generated by U_i^m

$$\overline{\Sigma}_{ij}^m = C_{ij}^{kl}\overline{U}_{k,l}^m - \gamma\overline{\theta}^m\delta_{ij}; \quad \theta^m = U_{N+1}^m, \quad i,j,k,l = 1,2,\ldots,N, \quad m = 1,2,\ldots,N+1$$

$$\overline{G}_i^m(x,n,p) = \overline{\Sigma}_{ij}^m n_j, \quad \overline{G}_{N+1}^m(x,n,p) = (\overline{\theta}_{,j}^m(x,p) - \eta p \overline{U}_j^m(x,p))n_j$$

Σ is the stress tensor generated by pulsed actions; the tensor with components G_i^m for $i = 1, 2, \ldots, N$ describes the stresses on an area with normal n, and for $i = N + 1$ it describes the heat flux over this area combined with losses on volume expansion.

We will introduce the notation

$$O_\varepsilon(x) = \{y \in S; \| x - y \| < \varepsilon\}, \quad \mathrm{Sph}_\varepsilon(x) = \{y \in R^N; \ \| x - y \| < \varepsilon\}$$

$$\mathrm{Sph}_\varepsilon^\pm = \mathrm{Sph}_\varepsilon \cap S^\pm, S_\varepsilon = S - O_\varepsilon, \quad S_\varepsilon^- = S^- - \mathrm{Sph}_\varepsilon^-, \quad \Gamma_\varepsilon^\pm = \{y : \| x - y \| = \varepsilon, y \in S^\pm\}$$

according to sign; $x \in S$.

Theorem 5.2 (*the dynamic analogue of Gauss' formula*). If S is a Lyapunov surface, then

$$\text{v.p.} \int_S \overline{G}_i^m(x-y,n(y),p)dS(y) + \rho p^2 \int_{S^-} \overline{U}_i^m(x-y,p)dV(y) = \rho\delta_i^m H_S^-(x)$$

$$\text{v.p.} \int_S \overline{G}_{N+1}^m(x-y,n(y),p)dS(y) + p\varkappa^{-1} \int_{S^-} \overline{U}_{N+1}^m(x-y,p)dV(y) = \delta_{N+1}^m H_S^-(x) \tag{5.2}$$

Proof. We convolute the equations for Green's tensor

$$\overline{\Sigma}_{ij,j}^m(x,p) - p^2\rho\overline{U}_i^m + \delta_i^m\delta(x) = 0$$

$$\overline{\theta}_{,jj}^m - p\varkappa^{-1}\overline{\theta}^m - \eta p\overline{U}_{j,j}^m + \delta_{N+1}^m\delta(x) = 0$$

with $H_S^-(x)$. Taking into account the rules for the differentiation of a convolution and formulae (4.1), we obtain

$$-\overline{\Sigma}_{ij}^m(x,p) * n_j\delta_S(x) - p^2\rho\overline{U}_i^m * H_S^-(x) + \rho\delta_i^m H_S^-(x) = 0,$$

$$-\overline{\theta}_{,j}^m * n_j\delta_S(x) - p\varkappa^{-1}\overline{\theta}^m * H_S^-(x) + \eta p\overline{U}_j^m * n_j\delta_S(x) + \delta_{N+1}^m H_S^-(x) = 0$$

The integral representation of this convolution, taking into account the notation introduced, has the form of (5.2), and here all the integrals exist in the usual sense by virtue of the continuity of the integrands for $x \notin S$.

We will show that formulae (5.2) also hold for $x \notin S$ in the sense of Definition 2.1.

We transform the contour in the neighbourhood of the point x, circumventing it along the ε-circle Γ_ε^- to S^- and Γ_ε^+ to $S^+ = R_{N^-}(S^- + S)$. We put $S_\varepsilon^\pm = S^- \pm \text{Sph}_\varepsilon^\pm$ and write each of the equalities (5.2) for the regions S_ε^+ and S_ε^-.

$$A_i^m + B_i^{m+} + \rho p^2 C_i^{m+} = \delta_i^m, \quad A_i^m + B_i^{m-} + \rho p^2 C_i^{m-} = 0$$

$$A_{N+1}^m + B_{N+1}^{m+} + \frac{p}{\varkappa}C_{N+1}^{m+} = \delta_i^{N+1}, \quad A_{N+1}^m + B_{N+1}^{m-} + \frac{p}{\varkappa}C_{N+1}^{m-} = 0$$

$$A_j^m = \int_{S_\varepsilon} \overline{G}_j^m(x-y,n(y),p)dS(y), \quad B_j^{m\pm} = \int_{\Gamma_\varepsilon^\pm} \overline{G}_j^m(x-y,n(y),p)dS(y)$$

$$C_j^{m\pm} = \int_{S_\varepsilon^\pm} \overline{U}_j^m(x-y,p)dV(y), \quad j = 1,2,\ldots,N+1$$

We add the equations in each line pairwise and take the limit as $\varepsilon \to 0$. The volume integrals $C_j^{m\pm}$ approach the integral over the entire region by virtue of the weak singularity with respect to r of the integrand (see Table 1). We single out the parts of the tensor with components $\overline{G}_i^k$ that have strong singularity with respect to r:

$$\tilde{G}_i^m = C_{ij}^{kl}\overline{U}_{k,l}^m n_j, \quad \tilde{G}_{N+1}^m(x,n,p) = \overline{U}_{N+1,j}^m n_j, \quad m = 1,2,\ldots,N+1, i,j,k,l = 1,2,\ldots,N \tag{5.3}$$

Since $\tilde{G}_i^m(x, n, p) = -\tilde{G}_i^m(-x, n, p)$ and, at opposite points y^+ and y^- of the sphere $\text{Sph}_\varepsilon\, y^+ - x = -(y^- - x)$, $n(y^+) = n(y^-)$, we have

$$\lim_{\varepsilon\to 0}\{\overline{B}_j^{m+} + \overline{B}_j^{m-}\} = 0, \quad j,m = 1,2,\ldots,N+1$$

As a result we obtain the proof.

If in (5.2) it is assumed that $S^- = \{y : \|x-y\| < \varepsilon\}$ and use is made of the weak asymptotic form $\overline{U}_i^m(x,p)$ when $\|x\| \to 0$, we obtain the following corollary.

Corollary 5.1. For all x

$$\lim_{\varepsilon\to 0} \int_{r=\varepsilon} \overline{G}_i^m\left(x-y, \frac{y-x}{\varepsilon}, p\right)dS(y) = \delta_i^m$$

6. BOUNDARY INTEGRAL EQUATIONS OF BOUNDARY-VALUE PROBLEMS OF COUPLED THERMOELASTODYNAMICS

Theorem 6.1. If the classical solution $u(x, t)$ of the boundary-value problem exists, $\overline{u}(x, p) \in C(S^- + S)$, and satisfies the Hölder condition on S for any fixed $p(\mathrm{Re}\, p > 0)$, then

$$\overline{u}_m H_S^-(x) = \overline{U}_m^j * (\hat{F}_j + \hat{G}_j) + \int_S (\overline{U}_m^k(x-y,p)\overline{q}_k^S(y,p) + \overline{T}_m^k(x-y,p,n(y))\overline{u}_k^S(y,p))dS(y) \qquad (6.1)$$

$\overline{q}_{N+1}^s = \overline{q}^s(x,p), \overline{u}_{N+1}^s = \overline{\theta}^s(x,p)$. For $x \in S$, the singular integral is taken in the sense of the principal value.

Proof. We will return to the analogue of Somigliana's formula (2.7), the integral notation of which, by virtue of the singularity of the solution, has the form indicated. Since U_m^k and accordingly T_m^k (the solution of system (1.1)) also have singularities only when $x - y = 0$, i.e. on S, then the integrals on the right-hand side of Eq. (6.1) exist for all $x \notin S$. Consequently, (2.7) is the solution of system (1.1) in S^-.

To prove Eq. (6.1) on the boundary, we consider the asymptotic forms of $\overline{T}_m^k$ and $\overline{G}_k^m$ when $r \to 0$. By relations (5.1) , we have

$$\overline{T}_m^k(x,n,p) \sim C_{ij}^{kl}\overline{U}_{m,j}^i n_l, i,j,k,l = 1,2,\dots,N;\ \overline{T}_m^{N+1} \sim \overline{U}_{m,j}^{N+1} n_j,\ m = 1,2,\dots,N+1$$

From the properties of symmetry (3.7) it follows that

$$\overline{G}_k^m \sim C_{kj}^{il}\overline{U}_{i,l}^m n_j = C_{kl}^{ij}\overline{U}_{m,j}^i n_l = C_{ij}^{kl}\overline{U}_{m,j}^i n_l \sim \overline{T}_m^k$$

$$\overline{G}_{N+1}^{N+1} \sim \overline{U}_{N+1,j}^{N+1} n_j = \overline{T}_{N+1}^{N+1},\ m = 1,2,\dots,N,\ i,j,k,l = 1,2,\dots,N$$

Where $r \to 0$ we have

$$\overline{U}_{N+1}^k = O_N,\quad \overline{U}_k^{N+1} = O_N,$$

$$\overline{G}_{N+1}^j(x,n,p) \sim \overline{U}_{N+1,j}^k n_j = O_N,\quad \overline{G}_k^{N+1}(x,n,p) \sim C_{kl}^{ij}\overline{U}_{i,j}^{N+1} n_j O_N,$$

$$\overline{T}_{N+1}^k(x,n,p) = O_N,\quad \overline{T}_k^{N+1} = O_N,\quad k = 1,2,\dots,N$$

$$O_2 = O(\ln r),\quad O_3 = O(r^{-1})$$

This results in the equalities

$$\lim_{\varepsilon \to 0} \int_{r=\varepsilon} \overline{T}_i^m(x-y,n(y),p)dS(y) = \delta_i^m \qquad (6.2)$$

$$\lim_{\varepsilon \to 0} \int_{r=\varepsilon} \overline{T}_i^{N+1}(x-y,n(y),p)dS(y) = \delta_i^{N+1} \qquad (6.3)$$

which hold for $N = 2, 3$.

Suppose $x \in S$. We consider the region $S^- - \mathrm{Sph}_\varepsilon^-(x)$ with the boundary $S_\varepsilon + \Gamma_\varepsilon^-$. Writing the analogue of Somigliana's formula for this region, since $x \notin S^- - \mathrm{Sph}_\varepsilon^-$, we obtain

$$0 = \int_{S_\varepsilon} \overline{U}_i^k(x-y)f_k(y)dS_\varepsilon(y) + \int_{\Gamma_\varepsilon^-} \overline{U}_i^k(x-y)f_k(y)d\Gamma_\varepsilon^-(y) +$$

$$+ \int_{S_\varepsilon} \overline{T}_i^k(x-y,n(y),p)u_k(y)dS_\varepsilon(y) + \int_{\Gamma_\varepsilon^-} \overline{T}_i^k(x-y,n(y),p)u_k(y)d\Gamma_\varepsilon^-(y) \qquad (6.4)$$

Taking the limit as $\varepsilon \to 0$, we have

$$0 = \mathrm{v.p.}\int_S \left\{ \sum_{m=1}^{N+1} [\overline{T}_i^m(x-y,n(y),p)u_m(y) + \overline{U}_i^m(x-y)f_m(y)] \right\} dS(y) +$$

$$+ \lim_{\varepsilon \to 0} \int_{\Gamma_\varepsilon^-} \overline{T}_i^m(x-y,n'(y),p)u_m(y,p)dS(y),\quad n'(y) = \frac{x-y}{\|x-y\|}$$

The integral in the sense of the principal value exists by virtue of the indicated asymptotic form of the tensor with components $\overline{T}_i^m(x,p)$ and the properties of antisymmetry of the strong singularity. We will calculate the average limits using the Hölder condition and Eqs (6.2) and (6.3)

$$\lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon} \overline{T}_i^m(x-y,n'(y),p)u_m(y)dS(y)=\lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon} \overline{T}_i^m(x-y,n'(y),p)(u_m(y)-u_m(x))dS(y)+$$

$$+\lim_{\varepsilon\to 0}u_m(x)\int_{\Gamma_\varepsilon} \overline{T}_i^m(x-y,n'(y),p)dS(y)=u_m(x)\lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon}(\overline{T}_i^m(x-y,n'(y),p)-$$

$$-\overline{G}_m^i(x-y,n'(y),p))dS(y)+u_m(x)\lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon^-}\overline{G}_m^i(x-y,n'(y),p)dS(y)=-\frac{1}{2}u_m(x)$$

Transferring the last term of the equality to the left-hand side of Eq. (6.4), we obtain relation (6.1), where the integrals containing $\overline{T}_m^k$ are singular and taken in the sense of the principal value. The theorem is proved.

7. CONCLUSION

Formulae (6.3) for $x \in S$ are boundary integral equations that enable the four boundary-value problems to be solved. All the equations constructed lend themselves to a numerical solution by interpolation of the boundary and of the required functions by splines, the order of which is selected depending on the required accuracy of the solution of the problem. In the case of the first boundary-value problem, the algorithm of the numerical solution of the BIEs is well developed for solving similar kinds of static problems of the theory of elasticity. For the remaining boundary-value problems, the solution requires the use of different regularization methods.

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